

3 SEM TDC MTMH (CBCS) C 5

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(Nov/Dec)

MATHEMATICS

(Core)

Paper : C-5

(Theory of Real Functions)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

*The figures in the margin indicate full marks
for the questions*

1. (a) State the divergence criteria of a limit
of a function. 1+1=2

(b) Define cluster point of a set with an
example. 1+1=2

(c) Use ϵ - δ definition to establish that

$$\lim_{x \rightarrow c} x^2 = c^2$$

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(d) Let $f : A \rightarrow \mathbb{R}$ where $A \subseteq \mathbb{R}$ and $c \in \mathbb{R}$, a cluster point of A . Show that if f has a limit, when $x \rightarrow c$, then f is bounded. 3

(e) Let $f : A \rightarrow \mathbb{R}$ where $A \subseteq \mathbb{R}$ and $c \in \mathbb{R}$, a cluster point of A . If $a \leq f(x) \leq b$, $\forall x \in A$ and $x \neq c$, and $\lim_{x \rightarrow c} f(x)$ exists, then show that

$$a \leq \lim_{x \rightarrow c} f(x) \leq b \quad 3$$

(f) State and prove squeeze theorem. 1+3=4

(g) Show by using definition that

$$\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty \quad 3$$

(h) Let $f : A \rightarrow \mathbb{R}$ where $A \subseteq \mathbb{R}$ and $c \in A$. Then establish any one of the following : 3

(i) If f is continuous at $c \in A$, then given any ε -neighbourhood $V_\varepsilon(f(c))$ of $f(c)$, \exists a δ -neighbourhood $V_\delta(c)$ of c , such that if $x \in A \cap V_\delta(c)$, then

$$f(x) \in V_\varepsilon(f(c)).$$

$$V_\varepsilon(f(c)) = (f(c) - \varepsilon, f(c) + \varepsilon)$$

(ii) Let given any ϵ -neighbourhood $V_\epsilon(f(c))$ of $f(c)$, \exists a δ -neighbourhood $V_\delta(c)$ of c , such that if $x \in A \cap V_\delta(c)$, then $f(x) \in V_\epsilon(f(c))$. Then f is continuous at $c \in A$.

(i) Let $f : A \rightarrow \mathbb{R}$ where $A \subseteq \mathbb{R}$ and define $|f|$ by $(|f|)(x) = |f(x)|$, $\forall x \in A$. Show that if f is continuous at $c \in A$, then $|f|$ is also continuous at $c \in A$.

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Or

Let $f : A \rightarrow \mathbb{R}$ where $A \subseteq \mathbb{R}$ and $f(x) \geq 0$, $\forall x \in A$. Defined \sqrt{f} by $(\sqrt{f})(x) = \sqrt{f(x)}$, $\forall x \in A$. Show that if f is continuous at $c \in A$, then \sqrt{f} is continuous at c .

(i) State and prove location roots theorem.

1+3=4

Or

Let I be a closed and bounded interval, and $f : I \rightarrow \mathbb{R}$ is continuous on I . Then show that $f : I \rightarrow \mathbb{R}$ is uniformly continuous.

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2. (a) Define relative maximum of a real-valued function at a point. 1
- (b) State the first derivative test for the relative maximum at a point of a real-valued function. 1
- (c) Show that if $f : I \rightarrow \mathbb{R}$ is differentiable and $f'(x) \geq 0, \forall x \in I$, then f is increasing on I . 2
- (d) Using first derivative test, show that $f(x) = x^2$ has a minima at $x = 0$. 2
- (e) State and prove the interior extremum theorem. 3

Or

Let $f : I \rightarrow \mathbb{R}$ be differentiable at c . If $f'(c) < 0$, then show that

$$f(x) < f(c), \quad \forall x \in (c - \delta, c)$$

- (f) State and prove Caratheodory's theorem. 4

(5)

- (g) Use mean value theorem to show that if $f(x) = \sin x$ which is differentiable, $\forall x \in \mathbb{R}$, then

$$|\sin x - \sin y| \leq |x - y| \quad \forall x, y \in \mathbb{R} \quad 4$$

Or

Use mean value theorem to show that

$$-x \leq \sin x \leq x \quad \forall x \geq 0$$

- (h) State and prove the mean value theorem. 4

- (i) State and prove Darboux's theorem. 4

Or

Use mean value theorem to show that

$$e^x \geq 1+x \quad \forall x \in \mathbb{R}$$

and hence show that $e^\pi > \pi^e$.

3. (a) Define a convex function on an interval and give its geometrical interpretation.

$$1+1=2$$

- (b) Show that the function $f(x) = x^3$ has no relative extremum at $x = 0$. 2

- (c) Show that

$$f(x) = x + \frac{1}{x}; \quad x > 0$$

is a convex function. 3

- (d) Determine relative extrema of the function

$$f(x) = x^4 + 2x^3 - k$$

where k is a constant. 3

- (e) State and prove Cauchy's mean value theorem. 5

- (f) State and prove Taylor's theorem with Lagrange's form of remainder. 5

- (g) Define Taylor's and Maclaurin's series. Obtain Maclaurin's series for the function $\sin x$. 2+3=5

(7)

Or

Show that

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{\underline{2n}} \quad \forall x \in \mathbb{R}$$

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